Complex Numbers IV Cheat Sheet (A Level Only)

In this topic, we will make use of the results $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

De Moivre's Theorem

When raising complex numbers to a power, we can use De Moivre's Theorem on their modulus-argument form,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

where $n \in \mathbb{R}$.

Example 1: Using De Moivre's Theorem, simplify $z = (2-2i)^7$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Write z in modulus-argument form.	$\lvert z\rvert = \sqrt{(2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ $\arg(z) = -\arctan\left(\frac{2}{2}\right) = -\frac{\pi}{4}$ rad $\Rightarrow z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$
Use De Moivre's Theorem with $n = 7$ and convert back into Cartesian form.	$z^7 = (2\sqrt{2})^7\left(\cos\left(7\left(-\frac{\pi}{4}\right)\right) + i\sin\left(7\left(-\frac{\pi}{4}\right)\right)\right)$ $= 1024\sqrt{2}\left(\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right)$ $a = 1024\sqrt{2} \times \cos\left(-\frac{7\pi}{4}\right) = 1024$ $b = 1024\sqrt{2} \times \sin\left(-\frac{7\pi}{4}\right) = -1024$ $z^7 = 1024 + 1024i$

Euler's Relation

Euler's relation is stated as $e^{i\theta} = \cos\theta + i\sin\theta$. This relation can be used to express a complex number in exponential form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Example 2: Express $z = 2\left(\cos\left(\frac{\pi}{10}\right) - i\sin\left(\frac{\pi}{10}\right)\right)$ in exponential form.

Use $-\sin(x) = \sin(-x)$ and $\cos(x) = \cos(-x)$ to rewrite z.	$-\sin\left(\frac{\pi}{10}\right) = \sin\left(-\frac{\pi}{10}\right), \quad \cos\left(\frac{\pi}{10}\right) = \cos\left(-\frac{\pi}{10}\right)$ $\Rightarrow z = 2\left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)$
Identify r and θ and write in exponential form.	$r = 2, \quad \theta = -\frac{\pi}{10}$ $z = 2e^{-\frac{\pi}{10}i}$

We can apply De Moivre's Theorem to the exponential form to obtain,

$$z^n = r^n e^{ni\theta}$$

Example 3: Given that $z = 4e^{\frac{\pi}{8}i}$ and $w = 2e^{\frac{\pi}{4}i}$, find $\left(\frac{z}{w}\right)^{10}$.

Use laws of indices to simplify $\frac{z}{w}$	$\frac{z}{w} = \frac{4e^{\frac{\pi}{8}i}}{2e^{\frac{\pi}{4}i}} = \left(\frac{4}{2}\right)e^{\left(\frac{\pi}{8}-\frac{\pi}{4}\right)i} = 2e^{-\frac{\pi}{8}i}$
Use De Moivre's Theorem with $n = 10$ to find $\left(\frac{z}{w}\right)^{10}$.	$\left(\frac{z}{w}\right)^{10} = (2)^{10}e^{-\frac{10\pi}{8}i} = 1024e^{-\frac{5\pi}{4}i}$

Complex conjugates can also be expressed in the exponential form,

$$z = re^{i\theta}, \quad z^* = re^{-i\theta}$$

Using De Moivre's Theorem to find Multiple Angle Identities

De Moivre's Theorem can be used to derive multiple angle identities in two main ways:

- Expressing arguments in terms of powers: $\cos(n\theta) = \cos^n\theta + \cos^{(n-2)}\theta + \cdots$
- Expressing powers in terms of arguments: $\cos^n\theta = \cos(n\theta) + \cos((n-2)\theta) + \cdots$

Example 4: Use De Moivre's Theorem to express $\sin3\theta$ in terms of powers of $\sin\theta$.

First, we use De Moivre's Theorem on $\cos4\theta + i\sin4\theta$ and expand using the binomial theorem. A useful pattern to note is that the expansion alternates between an i term and non-i term and follows a $+, +, -, -$ pattern for the sign of the coefficients. We simplify the working out by using C to represent $\cos\theta$ and S to represent $\sin\theta$.	$(\cos3\theta + i\sin3\theta) = (\cos\theta + i\sin\theta)^3$ $= C^3 + 3iC^2S + 3i^2CS^2 + i^3S^3$ $= C^3 + 3iC^2S - 3CS^2 - iS^3$ Equating Imaginary parts, $\sin3\theta = 3C^2S - S^3$
We can use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to remove the C^2 term.	$C^2 = 1 - S^2$ $\Rightarrow 3C^2S - S^3 = 3S(1-S^2) - S^3$ $= 3S - 3S^3 - S^3 = 3S - 4S^3$ $\sin3\theta = 3\sin\theta - 4\sin^3\theta$

To use the second method, we must first know two important trigonometric identities that can be derived using Euler's relation:

$$\sin n\theta = \frac{1}{2i}\left(e^{ni\theta} - e^{-ni\theta}\right), \qquad \cos n\theta = \frac{1}{2}\left(e^{ni\theta} + e^{-ni\theta}\right)$$

Example 5: Express $\cos^5\theta$ in terms of $\cos n\theta$.

We must make use of the identity $\cos n\theta = \frac{1}{2}\left(e^{ni\theta} + e^{-ni\theta}\right)$ and the binomial theorem.	Let $z = e^{i\theta}$, $\cos^5\theta = (\cos\theta)^5 = \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^5$ $= \frac{1}{32}(z + z^{-1})^5$ $= \frac{1}{32}(z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5})$ $= \frac{1}{32}(z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5})$
Group together "like" powers and use $2\cos n\theta = \left(e^{ni\theta} + e^{-ni\theta}\right)$.	$\frac{1}{32}((z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}))$ $= \frac{1}{32}(2\cos5\theta + 10\cos3\theta + 20\cos\theta)$ $= \frac{1}{16}\cos5\theta + \frac{5}{16}\cos3\theta + \frac{5}{8}\cos\theta$

Sum of Binomial Series

To sum binomial series, we must know two results that can be derived using Euler's relation and double angle formulae:

$$1 + e^{i\theta} = 2\cos\left(\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}, \qquad 1 - e^{i\theta} = -2i\sin\left(\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}$$

Example 6: Show that $1 + e^{i\theta} = 2\cos\left(\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}$. Use this result to find the sum of the series $1 + 3\cos\theta + 3\cos2\theta + \cos3\theta$.

We must make use of Euler's relation and the double angle identities $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.	$1 + e^{i\theta} = 1 + \cos\theta + i\sin = 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = 2\cos\frac{\theta}{2}e^{\frac{i\theta}{2}}$
Next, we identify that the binomial series presented is the real part of $\left(1 + e^{i\theta}\right)^3$.	$\left(1 + e^{i\theta}\right)^3 = \left(2\cos\frac{\theta}{2}\right)^3 e^{\frac{3i\theta}{2}} = 8\cos^3\frac{\theta}{2}e^{\frac{3i\theta}{2}}$
The two expressions for the real part of $\left(1 + e^{i\theta}\right)^3$ are equated.	$1 + 3\cos\theta + 3\cos2\theta + \cos3\theta = Re\left(8\cos^3\frac{\theta}{2}e^{\frac{3i\theta}{2}}\right)$ $= Re\left(8\cos^3\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)\right)$ $= 8\cos^3\frac{\theta}{2}\cos\frac{3\theta}{2}$

Sum of Finite and Infinite Geometric Series

The sum of a finite geometric series is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

where a is the first term of the series, r is the common ratio and n is the number of terms.

Example 7: Given two series $P = 1 + \cos3\theta + \cos6\theta + \cdots + \cos15\theta$ and $Q = \sin3\theta + \sin6\theta + \cdots + \sin15\theta$, find an expression for P in terms of sin and cos.

Combine the two series in the form $P + iQ$. This forms a geometric series. Identify a, r and n to use $S_n = \frac{a(1-r^n)}{1-r}$.	$P + iQ = 1 + \cos3\theta + \sin3\theta + \cos6\theta + \sin6\theta + \cdots$ $= 1 + e^{3i\theta} + e^{6i\theta} \ldots$ $a = 1, r = e^{3i\theta}, n = 6$ $\frac{\left(1 - (e^{3i\theta})^6\right)}{1 - e^{3i\theta}} = \frac{\left(1 - e^{18i\theta}\right)}{1 - e^{3i\theta}}$
Multiply the numerator and denominator by $e^{-\frac{3}{2}i\theta}$ (the complex conjugate of $e^{\frac{3}{2}i\theta}$).	$\frac{\left(1 - e^{18i\theta}\right)}{1 - e^{3i\theta}} \times \frac{e^{-\frac{3}{2}i\theta}}{e^{-\frac{3}{2}i\theta}} = \frac{\left(e^{-\frac{3}{2}i\theta} - e^{\frac{33}{2}i\theta}\right)}{e^{-\frac{3}{2}i\theta} - e^{\frac{3}{2}i\theta}}$ $= \frac{\left(e^{\frac{33}{2}i\theta} - e^{-\frac{3}{2}i\theta}\right)}{e^{\frac{3}{2}i\theta} - e^{-\frac{3}{2}i\theta}}$
Factor out $e^{\frac{15}{2}i\theta}$ from the numerator so that $e^{ni\theta} - e^{-ni\theta} = 2i\sin n\theta$ can be used.	$\frac{e^{\frac{15}{2}i\theta}\left(e^{9i\theta} - e^{-9i\theta}\right)}{e^{\frac{3}{2}i\theta} - e^{-\frac{3}{2}i\theta}} = \frac{e^{\frac{15}{2}i\theta}(2i\sin9\theta)}{2i\sin\frac{3}{2}\theta}$ $= \left(\cos\frac{15}{2}\theta + i\sin9\theta\right)\left(\frac{\sin9\theta}{\sin\frac{3}{2}\theta}\right)$ Equating Real Parts, $P = \cos\frac{15}{2}\theta\left(\frac{\sin9\theta}{\sin\frac{3}{2}\theta}\right)$

The sum of an infinite geometric series is given by,

$$S_\infty = \frac{a}{1-r}$$

where a is the first term of the series and r is the common ratio. We require the series to be convergent so $\lvert r\rvert < 1$.

Example 8: Given a convergent infinite geometric series $P = 1 + \frac{1}{4}e^{i\theta} + \frac{1}{16}e^{2i\theta} + \cdots$, find the sum to infinity of P. Hence, find the sum of the infinite series $C = 1 + \frac{1}{4}\cos\theta + \frac{1}{16}\cos2\theta + \cdots$.

Identify a and r, then use $S_\infty = \frac{a}{1-r}$.	$a = 1, \quad r = \frac{1}{4}e^{i\theta}$ $S_\infty = \frac{1}{1 - \frac{1}{4}e^{i\theta}} = \frac{4}{4 - e^{i\theta}}$
Multiply the numerator and denominator by $4 - e^{-i\theta}$ since $e^{-i\theta}$ is the complex conjugate of $e^{i\theta}$ in the denominator.	$\frac{4}{4 - e^{i\theta}} \times \frac{4 - e^{-i\theta}}{4 - e^{-i\theta}} = \frac{16 - 4e^{-i\theta}}{16 - 4e^{i\theta} - 4e^{-i\theta} + 1}$ $= \frac{16 - 4e^{-i\theta}}{17 - 4(e^{i\theta} + e^{-i\theta})}$
Use Euler's relation and $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$ to rewrite the fraction in terms of cos and sin. Also use the results $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$. C is the real part of $\frac{4}{4 - e^{i\theta}}$.	$\frac{16 - 4(\cos(-\theta) + i\sin(-\theta))}{17 - 8\cos\theta}$ $= \frac{16 - 4(\cos\theta - i\sin\theta)}{17 - 8\cos\theta} = \frac{16 - 4\cos\theta + 4i\sin\theta}{17 - 8\cos\theta}$ $C = Re\left(\frac{16 - 4\cos\theta + 4i\sin\theta}{17 - 8\cos\theta}\right) = \frac{16 - 4\cos\theta}{17 - 8\cos\theta}$